A FAST METHOD FOR PREDICTION OF CROSSLINK LOSS AND ACQUISITION

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ABSTRACT

The establishment of crosslinks between neighboring satellites in a constellation generally mandates certain geometric constraints for their relative positions. A neighbor satellite in an adjacent plane must meet certain range, azimuth and elevation conditions, relative to the "observer" satellite, before the crosslink can be established, and the links must be shut off when the constraints are no longer met.

The problem of calculating when to turn the links on and off, and of determining where the observer is to look for its neighbor, is commonly solved by periodically feeding each satellite ephemeredes of all its neighbors or, alternatively, orbit states of its neighbors, which it can then propagate to any desired time. Of course, a full ephemeris for, say, four neighbors over several days would take up considerable memory in an on-board computer, while a full propagation of the orbit states can be computationally expensive.

A much simpler and faster analytic method is presented here. Acquisition and loss of crosslinks may be predicted by solving a quadratic equation and following a simple Newton-Raphson iteration scheme. In tests run on a SPARC 30 workstation, nine days' worth of crosslink events were predicted to within less than a second for each of four neighbors in less than 30 seconds. The position of a neighboring vehicle, in range, azimuth and elevation (along with their derivatives) can be found as a closed-form expression in time so that numerical propagation is unnecessary.

INTRODUCTION

The Iridium Satellite Constellation comprises 66 satellites in Mission orbit and 14 "spare" satellites in Storage orbits. All are maintained in near-polar, near-circular frozen orbits. Each Mission satellite communicates with its four nearest neighbors (fore, aft, right and left) via crosslinks during normal operations, and crosslinks may be established among some or all of the Storage satellites for testing purposes. The fore and aft neighbors are in the same orbit plane as the "observer" satellite and thus, because of the near-circularity of the orbits, move very little with respect to the observer. These crosslinks are in continuous operation. The right and left neighbors, however, are in different orbit planes and therefore their range, azimuth and elevation with respect to the observer are strong functions of time. These links are necessarily broken as the satellites approach the poles (or more precisely as they approach the point of intersection of their respective orbit planes) and must be reestablished some time after they have passed the intersection point. The reacquisition is generally based on some criteria that must be met for the range, azimuth and elevation, and the satellite software must be able to predict when and where, on the basis of said criteria, to look for the neighbors for reacquisition. The utility of the constellation depends upon the satellites' ability to do this reliably and efficiently.

A common method of giving the observer satellite the information required to reacquire its neighbor is simply to upload an ephemeris table for the neighbor to the observer. The generation of such a table on the ground can be computationally expensive and, depending on the frequency of upload, the table may need to be quite sizeable; hence uploading and storing it on board may be awkward or undesirable. Alternatively, the observer could be given a neighbor's position at some epoch, and it could propagate the relative position through time, using a search algorithm to locate the times at which the appropriate criteria are met. This method can be computationally expensive on-board, which may also be undesirable.

It turns out that the relative motion between the neighbor and the observer is well-approximated (to within meters) by a simple Fourier expansion in the mean anomaly, employing no terms of frequency larger than four times the orbit frequency. Closed-form expressions for the range, azimuth and elevation can be derived, along with their derivatives and other related quantities. The goal in this paper is to derive these closed-form equations and to show how they can be used to predict the acquisition and loss times of crosslinks.

GEOMETRY & CALCULATIONS

For the sake of conceptual simplicity, let us first imagine two satellites in circular, Keplerian orbits at the same altitude above the Earth, as in Figure 1. The inclinations of the orbits can be arbitrary, as can the separation of their nodes and phasing (i.e. separation of relative positions on their respective orbits). The planes of these two orbits are separated by an angle α, which can be found from spherical trigonometry as satisfying the equations:

$$\cos\alpha = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos\Delta\Omega \qquad\qquad \sin\alpha = \frac{\sin\Delta\Omega \sin i_2}{\sin u_1} \tag{1}$$

Hence, if we measure from the ascending nodes we can see that the sign of $\Delta\Omega$ gives the sign of α. It is already well-known that the line-of-sight vector between any two such satellites describes an ellipse in inertial space, i.e. the neighbor moves in an ellipse about the observer (References 1-4). However most satellites, Iridium satellites included, do not occupy an inertial reference frame; the oblong shape of the satellite bus and the resulting gravity gradient on the vehicle force the satellite's frame to rotate at the orbital angular speed. It is the relative motion in this new frame we are interested in.

Since our goal is to describe the line-of-sight vector from the observer satellite in orbit 1 to the neighbor satellite in orbit 2, in the frame of reference of the observer, let us first rotate the coordinate system about the center of the Earth so as to make the observer's orbital plane the "equatorial plane," as shown in Figure 2. In this system, the neighbor's "inclination" is simply α, and the system is oriented so that the x-axis is defined by the intersection of the orbit planes, the z-axis by the observer's angular momentum vector, and the y-axis is chosen to make up a right-handed Cartesian system. So, in terms of the satellite's common radial distance from the center of the Earth, *r*, and their true anomalies as measured from the x-axis, f_1 and f_2, we can write the line-of-sight vector as:

$$\vec{r} = r\begin{bmatrix} \cos f_2 - \cos f_1 \\ \cos\alpha \sin f_2 - \sin f_1 \\ \sin\alpha \sin f_2 \end{bmatrix}. \tag{2}$$

Now we can rotate the coordinate system again through an angle f about the z-axis so that the x-axis goes through the observer satellite. This gives us a relative position vector of

$$\vec{r}\,' = r\begin{bmatrix} \cos f_1 & \sin f_1 & 0 \\ -\sin f_1 & \cos f_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos f_2 - \cos f_1 \\ \cos\alpha \sin f_2 - \sin f_1 \\ \sin\alpha \sin f_2 \end{bmatrix} = r\begin{bmatrix} \cos f_1 \cos f_2 + \cos\alpha \sin f_1 \sin f_2 - 1 \\ \cos\alpha \cos f_1 \sin f_2 - \sin f_1 \cos f_2 \\ \sin\alpha \sin f_2 \end{bmatrix} \tag{3}$$

where the true anomalies are still measured from the orbits' point of intersection. Finally, we can convert to the SV body-fixed frame of reference (where the x-axis is in the direction of motion, the z-axis points to the center of the Earth, and the y-axis is in the direction of negative angular momentum) by the exchange of coordinates $x \rightarrow -z$, $y \rightarrow x$, $z \rightarrow -y$ so that

$$\vec{r}\,'' = r\begin{bmatrix} \cos\alpha \cos f_1 \sin f_2 - \sin f_1 \cos f_2 \\ -\sin\alpha \sin f_2 \\ 1 - \cos f_1 \cos f_2 - \cos\alpha \sin f_1 \sin f_2 \end{bmatrix}. \tag{4}$$

Now, because we have assumed that the satellites all travel in circular orbits, the line-of-sight vector can be expressed as a simple function of time. If the mean motion of the satellites is denoted by *n* and the (constant) difference in the true anomalies by Δf, we can write $f_1 = nt$, $f_2 = nt + \Delta f$ so that

$$\vec{r}\,'' = r\begin{bmatrix} \cos^2\frac{a}{2}\sin\Delta f - \sin^2\frac{a}{2}\cos\Delta f\,\sin 2nt - \sin^2\frac{a}{2}\sin\Delta f\,\cos 2nt \\ -\sin a\,\cos\Delta f\,\sin nt - \sin a\,\sin\Delta f\,\cos nt \\ 1 - \cos^2\frac{a}{2}\cos\Delta f - \sin^2\frac{a}{2}\cos\Delta f\,\cos 2nt + \sin^2\frac{a}{2}\sin\Delta f\,\sin 2nt \end{bmatrix}. \tag{5}$$

Notice that this vector, which represents the motion of the neighbor with respect to the observer, is actually a superposition of two very simple motions: one is a circular motion in the x-z plane with radius $r\sin^2\frac{a}{2}$ and period $\frac{p}{n}$, about the point $x = r\cos^2\frac{a}{2}\sin\Delta f$, $z = r\left(1-\cos^2\frac{a}{2}\cos\Delta f\right)$, and the other is a sinusoidal oscillation in the perpendicular direction with amplitude $r\sin a$ and period $\frac{2p}{n}$. The result is a kind of corkscrew path, which doubles back on itself to form a closed loop. If the neighbor satellite is in the same orbit plane as the observer, however, the motion degenerates to a single fixed point in the x-z plane.

It is useful and natural to look at the relative motion in spherical coordinates. With azimuth defined as the angle in the x-y plane, measured from the +x-axis toward the +y-axis, and elevation as the angle measured from the x-y plane toward the +z-axis, the range, elevation and azimuth of the neighbor satellite, respectively, are

$$\begin{aligned} R &\equiv \sqrt{\vec{r}\,''\circ\vec{r}\,''} \\ &= r\sqrt{2\left(1-\cos^2\frac{a}{2}\cos\Delta f - \sin^2\frac{a}{2}\cos\Delta f\,\cos 2nt + \sin^2\frac{a}{2}\sin\Delta f\,\sin 2nt\right)} \\ e &\equiv \sin^{-1}\left[\frac{z}{\sqrt{x^2+y^2}}\right] \\ &= \sin^{-1}\left[\sqrt{\frac{1-\cos^2\frac{a}{2}\cos\Delta f - \sin^2\frac{a}{2}\cos\Delta f\,\cos 2nt + \sin^2\frac{a}{2}\sin\Delta f\,\sin 2nt}{2}}\right] \qquad (6) \\ Az &\equiv \tan^{-1}\left[\frac{y}{x}\right] \\ &= \tan^{-1}\left[\frac{-\sin a\left(\cos\Delta f\,\sin nt + \sin\Delta f\,\cos nt\right)}{\cos^2\frac{a}{2}\sin\Delta f - \sin^2\frac{a}{2}\cos\Delta f\,\sin 2nt - \sin^2\frac{a}{2}\sin\Delta f\,\cos 2nt}\right]. \end{aligned}$$

From these coordinates we may easily derive range rate, distances of closest and farthest approach, angular rates and other important quantities.

Usually one thinks of the satellite phasing in their respective orbits in terms of the difference in true argument of latitude, mean argument of latitude or true anomaly, and not in terms of the angular distance from the point of intersection of the two planes. Therefore we relate *Df* to the difference in true argument of latitude (*Du*) through Napier's analogy:

$$\Delta f = \Delta u + 2\tan^{-1}\left[\tan\left(\frac{\Delta\Omega}{2}\right)\frac{\cos\left(\frac{i_1+i_2}{2}\right)}{\cos\left(\frac{i_1-i_2}{2}\right)}\right] \tag{7}$$

and use the value thus derived in Equations (6).

Although we have so far only looked at circular, common-altitude orbits, most satellite constellations do use frozen orbits, which are very nearly circular, at a common altitude. Therefore, the preceding examination of the results thus far gives a good qualitative idea of how the motion of the neighbor satellite looks to the observer satellite.

Now the very simple picture derived so far must be enhanced somewhat. Frozen orbits are not exactly circular and they are not quite Keplerian. But the addition of a small eccentricity does not affect the conceptual framework, except to distort slightly the path of the neighbor. The distortion is represented mathematically as a power series in eccentricity, so that we write:

$$\vec{r}'' = \vec{A}(t) + e\vec{B}(t) + e^2\vec{C}(t) + ... \tag{8}$$

Obviously, $\vec{A}$ is the solution we have already found (except that the zero of time can now be unambiguously defined as the time of the observer's perigee passage), while the other vectors are found by retracing the above derivation and using Kepler's Equation to relate the true anomalies of the satellites to the time. How far to extend the series depends on the accuracy required. For a frozen orbit at the upper reaches of LEO $e \approx 0.0013$, so that truncation at the first power of eccentricity suffices to give 10-meter accuracy, whereas inclusion of e^2 ensures sub-meter accuracy. The vectors used in Equation (8), all simple trigonometric functions of time, are given explicitly in the Appendix. Each of the quantities therein has to be calculated only once, and then the line-of-sight vector and all related quantities can be found easily at any desired epoch.

As for perturbations, the most important are those due to J_2, J_3, J_4 and J_{22}. For frozen orbits, it happens that J_2 is of the same order of magnitude as the eccentricity, while J_3, J_4 and J_{22} are all of order e^2. So we could expand the line-of-sight vector yet again, taking all these terms into account, but in most cases this is unnecessary. The purpose and utility of satellite constellations generally depend upon the uniform, repeating relative motions among their satellites. Therefore the satellites are almost always placed in common-inclination orbits, so that their RAAN rates are synchronized. Thus with common inclinations, common semi-major axes, common eccentricities and arguments of perigee (for frozen orbits), the effects of all the zonal terms in the geopotential on the satellites' mean elements drop completely out. This is an inevitable consequence of the lack of dependence of the zonal perturbations on RAAN, and the averaging of orbit elements over the mean anomaly.

To the extent that the inclinations are not all precisely identical, typical offsets among them (these can be as large as $0^{\circ}.005$) are between first and second order quantities which, when multiplied by J_2, may or may not be important to the analysis. An evaluation must be made as to whether the resulting terms are as large as second order and whether second order is even required in the analysis. In any case, the main factor is the growth in the RAAN difference between the two orbit planes:

$$d\Omega = \left[\frac{3}{2}\frac{nJ_2R_e^2}{a^2}\sin i\right]di\Delta t = K\Delta t \tag{9}$$

where K is a very small quantity (for Iridium orbits it is about $1.17x10^{-10}$ $\sec^{-1}$ for maximum offset). If it is deemed necessary to include this effect, one can either recalculate all the quantities in the Appendix by writing:

$$\begin{aligned} Q &= \sin(\Delta\Omega_o + K\Delta t)\frac{\cos i_{neighbor} - \cos i_{observer}}{2} \\ &\approx \sin\Delta\Omega_o \frac{\cos i_{neighbor} - \cos i_{observer}}{2} + K\cos\Delta\Omega_o \frac{\cos i_{neighbor} - \cos i_{observer}}{2}\Delta t \\ &\approx Q_o + Q_1\Delta t \end{aligned} \tag{10}$$

and so on for the other five "constants" and then following through to find the vector $\vec{A}$ from Equation (8) as the sum of two new vectors: $\vec{A}_o(M)+\vec{A}_1(M)\Delta t$ or by writing;

$$\begin{aligned}Q &= \sin(\Delta\Omega_o + K\Delta t)\frac{\cos i_{neighbor} - \cos i_{observer}}{2} \\ &= \sin\Delta\Omega_o \frac{\cos i_{neighbor} - \cos i_{observer}}{2}\cos(K\Delta t) + \cos\Delta\Omega_o \frac{\cos i_{neighbor} - \cos i_{observer}}{2}\sin(K\Delta t) \\ &= Q_o \cos(K\Delta t) + Q_1 \sin(K\Delta t)\end{aligned} \qquad (10)$$

and so on, and then ultimately folding the trigonometric terms into the original vector $\vec{A}$ to get terms in $kM \pm K\Delta t$ where there were originally terms in simply kM. Each of these "solutions" makes the calculations of the constants a bit more complicated and increases the number of terms in the $\vec{A}$ vector (note that the vectors $\vec{B}$ and $\vec{C}$ don't need to be adjusted because of their smaller size), but the additional inconvenience is still more than offset by the advantage of having closed-form expressions.

Herein lies the main strength of this method: the mean path traced out by each satellite with respect to the Earth is the Keplerian orbit plus [periodic and secular] terms of order e plus [periodic and secular] terms of order e^2, but the motion of one satellite with respect to another is simply the Keplerian relative motion plus [periodic and secular] terms or order e^2. This means that the relative motion vector can be determined for much longer periods of time than can the Earth-referenced vector, assuming the same accuracy requirements. Another strength of this method is that the line-of-sight vector can be determined at any point in time from an algebraic equation without the time-consuming operation of having to propagate it from the epoch to the time in question.

Numerical tests of the foregoing equations were made by propagating the mean orbit elements of Iridium satellites (or, more precisely, their control boxes, since routine station keeping is performed to maintain the vehicles close to their ideal positions) and comparing the resulting range, elevation and azimuth with the calculated values once per minute for 500 orbit periods. The values of the constants in the Appendix were used without modification for geopotential perturbations. Typical results are shown in Figures 3a-i for one-day periods starting at 0, 100 and 500 revs. The calculated range diverges from the "actual" (propagated) value by about three meters per revolution, maximum; this is indeed smaller than first order in the small quantities.

APPLICATION TO CROSSLINK ACQUISITION & LOSS

The criteria for maintenance of crosslinks between neighboring vehicles apparently vary somewhat among satellite constellations, as each constellation employs its own unique combination of satellite bus geometry, crosslink hardware and software, signal structure, etc. As has already been mentioned, the observer satellite's fore and aft neighbors hardly move with respect to the observer; therefore it is not difficult to design a constellation in which every vehicle maintains continuous crosslinks with its fore and aft neighbors. The challenge to simple, efficient design is in placing the left and right neighbors such that the constraints on their crosslink availability are minimized.

The Iridium constellation, for example, was designed so efficiently that whenever a neighbor satellite is farther away from the observer satellite than a certain minimum distance, all other criteria for crosslink acquisition and maintenance are automatically satisfied. Therefore the right and left crosslinks must be dropped at some specific time as the vehicles approach each other near their planes' point of intersection and they must be reacquired at a specific time as they move apart after that point. At the time of reacquisition, of course, the observer also needs to know where to "look for" the neighbor.

Note first that Equation (8) gives us the "exact" (to second order in small quantities) position of the neighbor as a function of time. We need an inverse equation, if we are to determine when a particular minimum range will be crossed. But this is not available in closed form, so we resort to Newton-Raphson iteration. We know the range of the neighbor satellite is given approximately by the first of Equations (6). That equation *can* be inverted in closed form, to give:

$$M = \frac{1}{2}\left\{\cos^{-1}\left[\frac{1-\cos^2\frac{a}{2}\cos\Delta f - \frac{1}{2}\left(\frac{R}{a}\right)^2}{\sin^2\frac{a}{2}}\right] - \Delta f\right\} \tag{11}$$

There are generally four possible values of *M* on any 360$^{\mathrm{o}}$ interval; these give the mean anomalies of the loss and reacquisition near the "North Pole" intersection of the planes and the corresponding points near the "South Pole."

With these approximate values in hand, we turn to Equation (8), which gives us an expression for the square of the range:

$$R^2 = \vec{A}\circ\vec{A} + 2e\vec{A}\circ\vec{B} + e^2\left(2\vec{A}\circ\vec{C} + \vec{B}\circ\vec{B}\right). \tag{12}$$

The approximate values of *M* from Equation (11) do not correspond exactly to the desired minimum crosslink range (we'll call it R_*), but give rather the approximate equality:

$$\Phi(M) \equiv \vec{A}\circ\vec{A} + 2e\vec{A}\circ\vec{B} + e^2\left(2\vec{A}\circ\vec{C} + \vec{B}\circ\vec{B}\right) - R_*^2 \approx 0\,. \tag{13}$$

An adjustment to the value of *M*, call it *dM*, can be found which, when added to *M*, corresponds better to the sought-for range, R_*, by setting:

$$\Phi(M + dM) = \Phi + \frac{\partial\Phi}{\partial M}dM = 0 \Rightarrow dM = -\frac{\Phi}{\partial\Phi/\partial M} \tag{14}$$

so that

$$dM = \frac{R_*^2 - \vec{A}\circ\vec{A} - 2e\vec{A}\circ\vec{B} - e^2\left(2\vec{A}\circ\vec{C} + \vec{B}\circ\vec{B}\right)}{2\left[\left(\vec{A} + e\vec{B} + e^2\vec{C}\right)\circ\frac{\partial\vec{A}}{\partial M} + e\left(\vec{A} + e\vec{B}\right)\circ\frac{\partial\vec{B}}{\partial M} + e^2\vec{A}\circ\frac{\partial\vec{C}}{\partial M}\right]}. \tag{15}$$

If this adjustment, when added to *M*, does not yield a value of *R* close enough to R_*, the process can be iterated until a satisfactory agreement is reached. Then at the end of the process, when the "true" value of *M* is known, reacquisition follows from calculating the azimuth and elevation from Equation (8) applied to the definitions of those quantities from Equations (6).

This algorithm was implemented on a SPARC 30 workstation to predict nine days' worth of crosslink loss and reacquisition times. Actual satellite data was used to calculate the constants in the Appendix, and a first estimate of the four points of interest was made from Equation (11) on the first revolution. Then the estimates were refined for each point according to Equation (15) until *R* was calculated to be within a meter of R_*. Then on each subsequent revolution, the final estimates from the previous rev were used as a first guess (same mean anomalies, times are exactly one orbit period later), and the iteration process proceeded from there. The entire process took less than 30 seconds, or about 20-40 times faster than the usual method of propagating ephemeredes for both vehicles and interpolating between ephemeris points for the events of interest. The two methods disagreed by no more than 0.96 seconds for any event on this interval, as can be seen from the time plot of the difference in Figure 4.

CONCLUSION

A closed-form expression for the line-of-sight vector between two neighboring vehicles in a satellite constellation has been derived, under the assumption that the satellites are in frozen orbits. The expression involves simple Fourier series and therefore makes it computationally inexpensive to find the relative position of one satellite with respect to another at any given epoch in rectangular or spherical coordinates.

The utility of this approach has been demonstrated in this paper by its application to the problem of determining the times of loss and reacquisition of crosslinks under a particular constraint. The extension of the method to different or additional constraints is straightforward.

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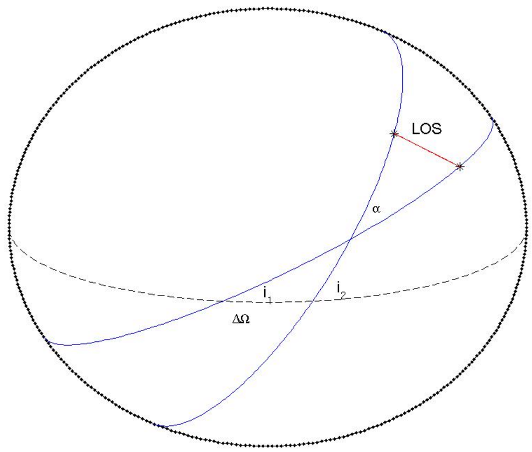


Figure 1: Schematic diagram of observer (Sat 1) and neighbor (Sat 2) satellites and relevant orbital parameters. The line-of-sight vector between the two is indicated.

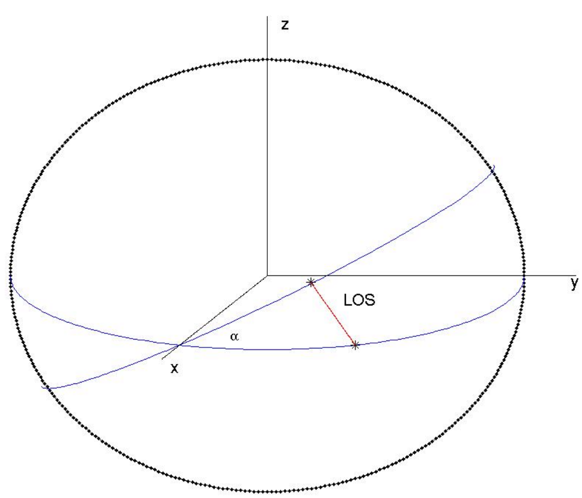
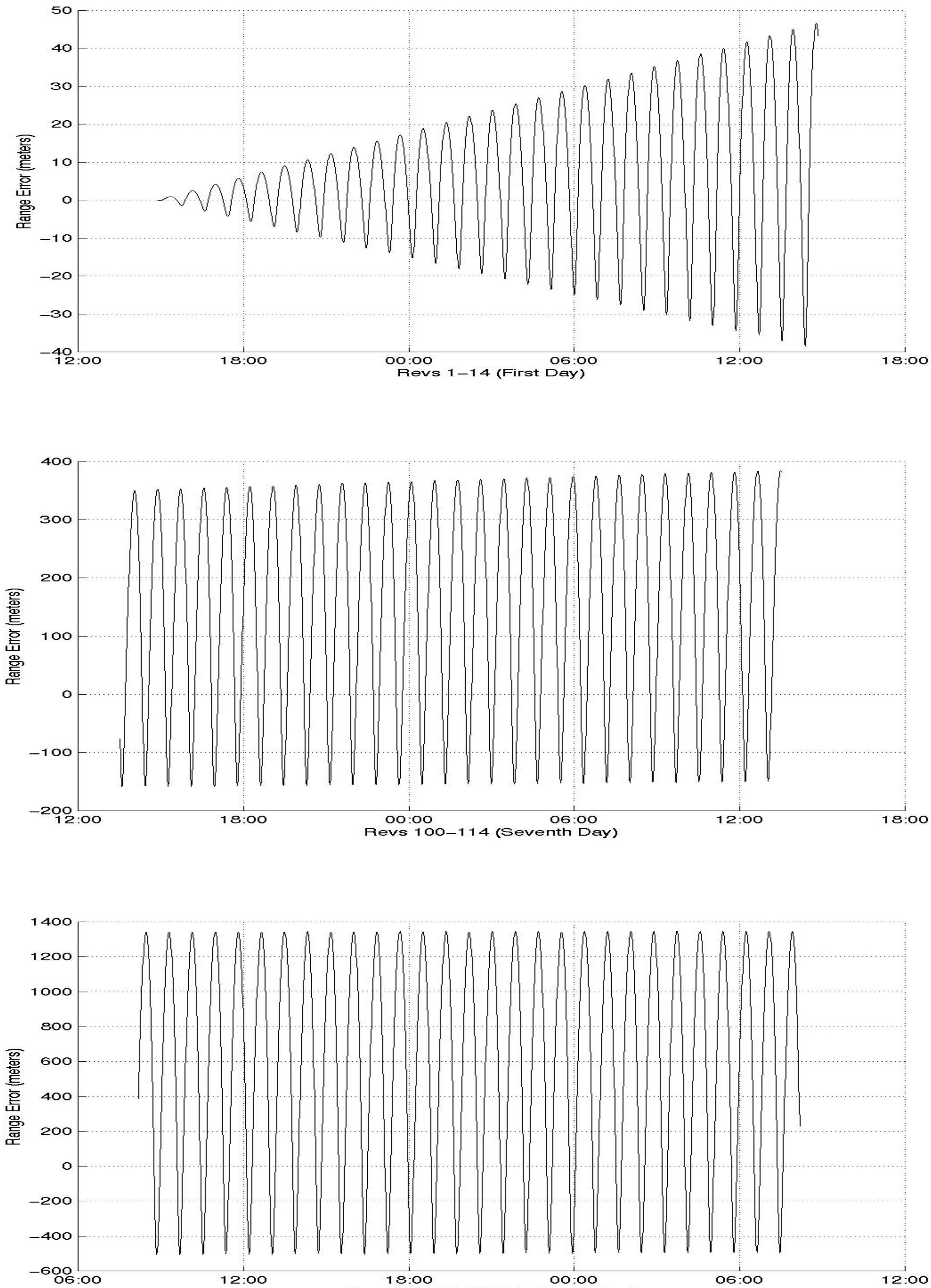
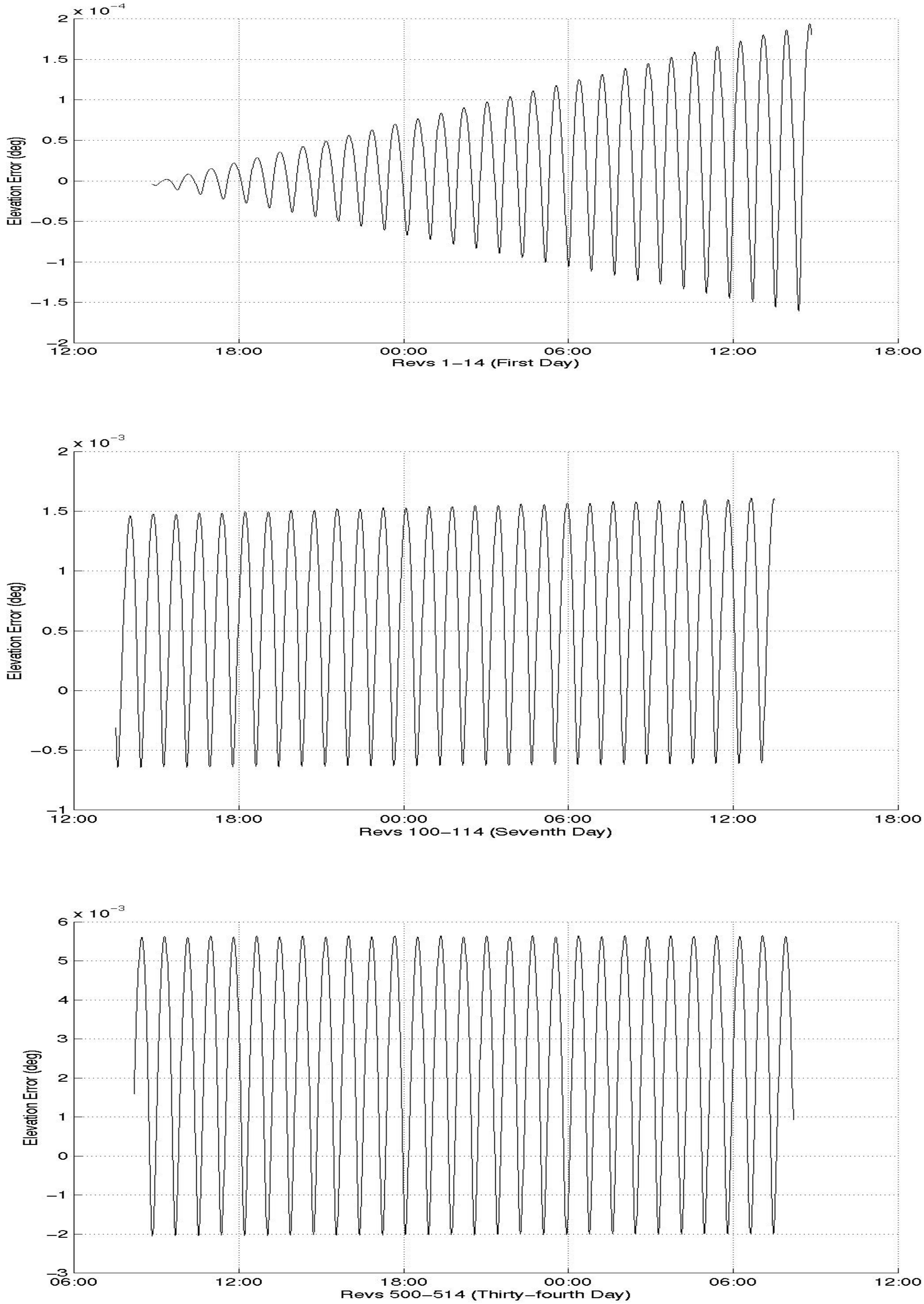


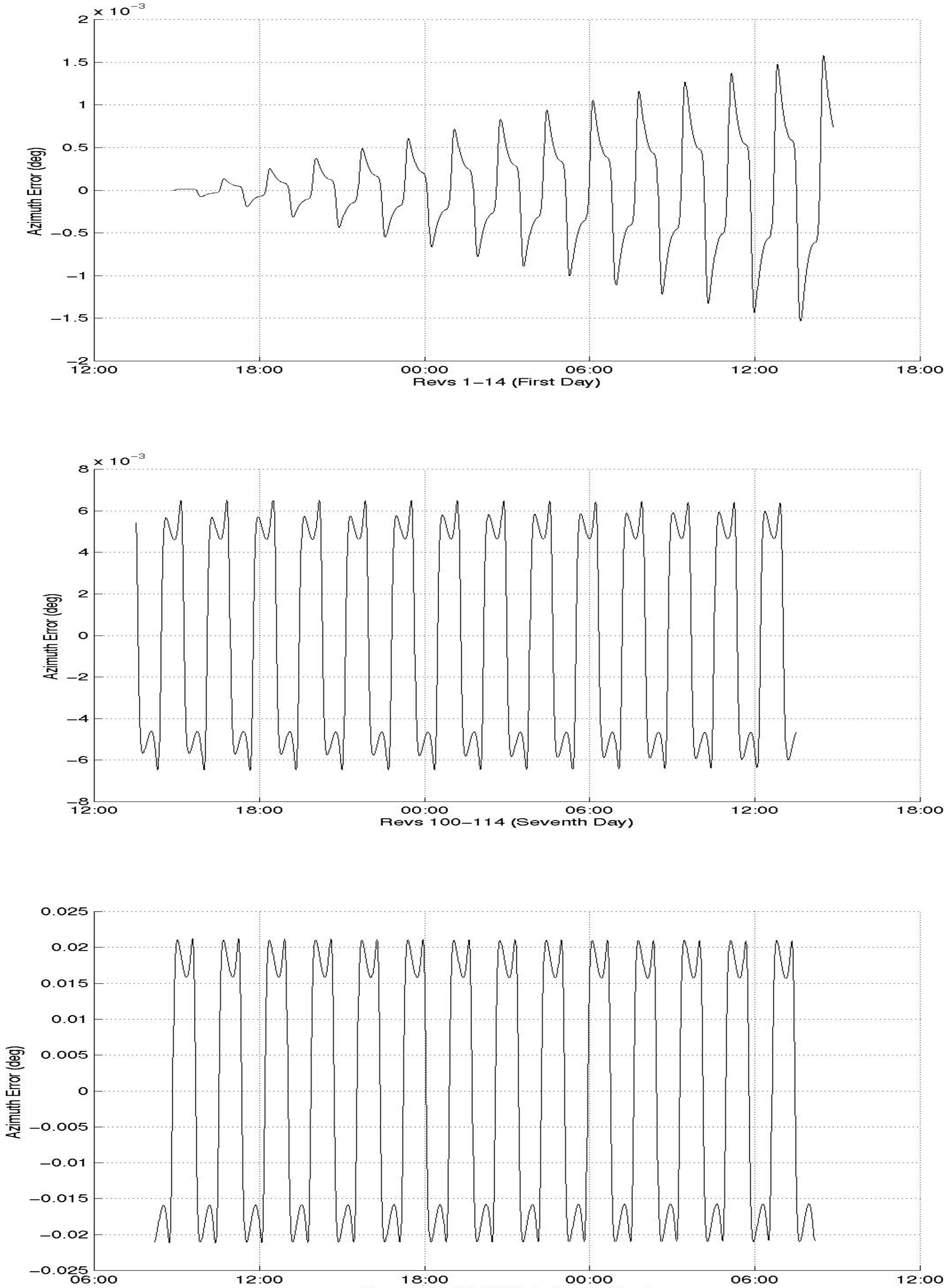
Figure 2: Motion of the observer and neighbor satellites in a frame of reference where the observer moves along the "equator." Inclination of neighbor's orbit in this frame is α.



Figures 3a-c: Growth of Range error (difference in Range between traditional method of ephemeris generation and calculation vs. analytic method presented here) over time.



Figures 3d-f: Growth of Elevation error (difference in Elevation between traditional method of ephemeris generation and calculation vs. analytic method presented here) over time.



Figures 3g-i: Growth of Azimuth error (difference in Azimuth between traditional method of ephemeris generation and calculation vs. analytic method presented here) over time.

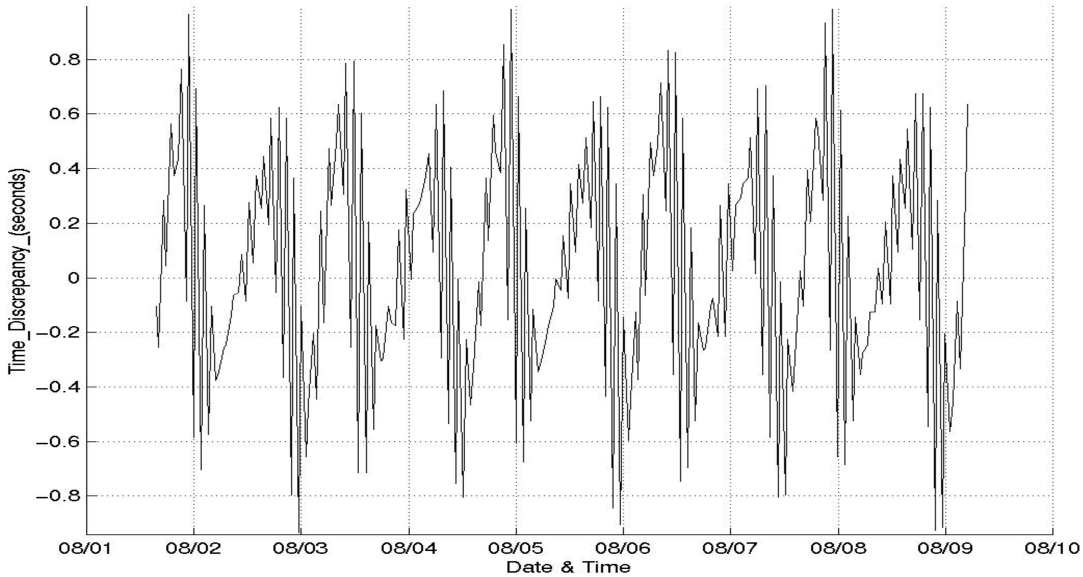


Figure 4: Error in Crosslink on/off times, algorithm vs. ephemeris, over nine days.

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APPENDIX

The expression for the line-of-sight vector between two satellites in common-altitude frozen orbits (argument of perigee frozen at 90^o) is given by

$$\vec{r}'' = \vec{A}(t) + e\vec{B}(t) + e^2\vec{C}(t) + O(e^3) \tag{A1}$$

where

$$\vec{A} = a\begin{bmatrix} A_{x1} + A_{x2}\sin 2M + A_{x3}\cos 2M \\ A_{y1}\sin M + A_{y2}\cos M \\ A_{z1} + A_{z2}\sin 2M + A_{z3}\cos 2M \end{bmatrix}$$

$$\vec{B} = a\begin{bmatrix} B_{x1}\sin M + B_{x2}\cos M + B_{x3}\sin 3M + B_{x4}\cos 3M \\ B_{y1} + B_{y2}\sin 2M + B_{y3}\cos 2M \\ B_{z1}\sin M + B_{z2}\cos M + B_{z3}\sin 3M + B_{z4}\cos 3M \end{bmatrix} \tag{A2}$$

$$\vec{C} = a\begin{bmatrix} C_{x1} + C_{x2}\sin 2M + C_{x3}\cos 2M + C_{x4}\sin 4M + C_{x5}\cos 4M \\ C_{y1}\sin M + C_{y2}\cos M + C_{y3}\sin 3M + C_{y4}\cos 3M \\ C_{z1} + C_{z2}\sin 2M + C_{z3}\cos 2M + C_{z4}\sin 4M + C_{z5}\cos 4M \end{bmatrix}$$

In these equations, M is the observer satellite's mean anomaly and is proportional to time ($M = nt = \frac{2pt}{P}$), *a* is the nominal semi-major axis, and the coefficients are calculated by first determining the constants:

$$\begin{aligned} Q &= \sin\Delta\Omega\frac{\cos i_{neighbor} - \cos i_{observer}}{2} \\ R &= \frac{\sin i_{neighbor}\sin i_{observer} - (1 - \cos i_{neighbor}\cos i_{observer})\cos\Delta\Omega}{2} \\ S &= \sin\Delta\Omega\frac{\cos i_{neighbor} + \cos i_{observer}}{2} \\ T &= \frac{\sin i_{neighbor}\sin i_{observer} + (1 + \cos i_{neighbor}\cos i_{observer})\cos\Delta\Omega}{2} \\ V &= \sin\Delta\Omega\sin i_{observer} \\ W &= \sin i_{neighbor}\cos i_{observer} - \cos i_{neighbor}\sin i_{observer}\cos\Delta\Omega \end{aligned} \tag{A3}$$

where $\Delta\Omega = \Omega_{neighbor} - \Omega_{observer}$. Then, with $\Delta f = f_{neighbor} - f_{observer}$ as the difference in *mean* argument of latitude, the coefficients are

$$\begin{aligned} A_{x1} &= S\cos\Delta f + T\sin\Delta f \\ A_{x2} &= -(R\cos\Delta f + Q\sin\Delta f) \\ A_{x3} &= -R\sin\Delta f + Q\cos\Delta f \\ A_{y1} &= W\sin\Delta f - V\cos\Delta f \\ A_{y2} &= -(W\cos\Delta f + V\sin\Delta f) \\ A_{z1} &= 1 - T\cos\Delta f + S\sin\Delta f \end{aligned}$$

$$A_{z2} = -A_{x3} \qquad A_{z3} = A_{x2}$$

$$B_{x1} = \frac{3}{2}(R+T) + (Q+2S)\sin\Delta f + (R-2T)\cos\Delta f + \frac{1}{2}(T\cos 2\Delta f - S\sin 2\Delta f)$$

$$B_{x2} = -\frac{3}{2}(S+Q) + R\sin\Delta f - Q\cos\Delta f + \frac{1}{2}(S\cos 2\Delta f + T\sin 2\Delta f)$$

$$B_{x3} = -(R\cos\Delta f + Q\sin\Delta f) - \frac{1}{2}(R\cos 2\Delta f + Q\sin 2\Delta f)$$

$$B_{x4} = Q\cos\Delta f - R\sin\Delta f + \frac{1}{2}(Q\cos 2\Delta f - R\sin 2\Delta f)$$

$$B_{y1} = \frac{3}{2}W$$

$$B_{y2} = \frac{W\sin 2\Delta f - V\cos 2\Delta f}{2}$$

$$B_{y3} = \frac{-(W\cos 2\Delta f + V\sin 2\Delta f)}{2}$$

$$B_{z1} = \frac{3}{2}(Q+S) + (Q-2S)\cos\Delta f - (R+2T)\sin\Delta f + \frac{1}{2}(S\cos 2\Delta f + T\sin 2\Delta f)$$

$$B_{z2} = -1 + \frac{3}{2}(R+T) + R\cos\Delta f + Q\sin\Delta f + \frac{1}{2}(S\sin 2\Delta f - T\cos 2\Delta f)$$

$$B_{z3} = -B_{x4} \qquad B_{z4} = B_{x3}$$

$$C_{x1} = \frac{3}{2}(S + Q - S\cos\Delta f - T\sin\Delta f) + \frac{1}{2}(S\cos 2\Delta f + T\sin 2\Delta f) + \frac{1}{4}R\sin\Delta f$$

$$C_{x2} = \frac{3}{2}(R + T + R\cos\Delta f + Q\sin\Delta f) + \frac{1}{2}(R-T)\cos 2\Delta f + \frac{1}{2}(S+Q)\sin 2\Delta f$$
$$+ \frac{3}{8}(T\cos 3\Delta f - S\sin 3\Delta f) + \frac{9}{8}S\sin\Delta f - \frac{11}{8}T\cos\Delta f$$

$$C_{x3} = -\frac{3}{2}(Q + S + Q\cos\Delta f - R\sin\Delta f) + \frac{1}{2}(T-R)\sin 2\Delta f + \frac{1}{2}(S+Q)\cos 2\Delta f$$
$$+ \frac{3}{8}(S\cos 3\Delta f + T\sin 3\Delta f) + \frac{9}{8}S\cos\Delta f + \frac{7}{8}T\sin\Delta f$$

$$C_{x4} = \frac{-R(9\cos\Delta f + 4\cos 2\Delta f + 3\cos 3\Delta f) - Q(9\sin\Delta f + 4\sin 2\Delta f + 3\sin 3\Delta f)}{8}$$

$$C_{x5} = \frac{-R(9\sin\Delta f + 4\sin 2\Delta f + 3\sin 3\Delta f) + Q(9\cos\Delta f + 4\cos 2\Delta f + 3\cos 3\Delta f)}{8}$$

$$C_{y1} = \frac{-3W\sin\Delta f + 5V\cos\Delta f}{8}$$

$$C_{y2} = \frac{3W\cos\Delta f + 5V\sin\Delta f}{8}$$

$$C_{y3} = \frac{3(W\sin 3\Delta f - V\cos 3\Delta f)}{8}$$

$$C_{y4} = \frac{-3(W\cos 3\Delta f + V\sin 3\Delta f)}{8}$$

$$C_{z1} = \frac{1-3(R+T+S\sin\Delta f - T\cos\Delta f) - T\cos 2\Delta f + S\sin 2\Delta f}{2} + \frac{1}{4}Q\sin\Delta f$$

$$C_{z2} = \frac{3}{2}(S+Q-R\sin\Delta f + Q\cos\Delta f) + \frac{1}{2}(Q-S)\cos 2\Delta f - \frac{1}{2}(R+T)\sin 2\Delta f$$

$$+\frac{3}{8}(S\cos 3\Delta f + T\sin 3\Delta f) - \frac{9}{8}T\sin\Delta f - \frac{11}{8}S\cos\Delta f$$

$$C_{z3} = \frac{3}{2}(R+T+R\cos\Delta f + Q\sin\Delta f) - \frac{1}{2}[1-R-T+(S-Q)\sin 2\Delta f]$$

$$+\frac{3}{8}(S\sin 3\Delta f - T\cos 3\Delta f) + \frac{7}{8}S\sin\Delta f - \frac{9}{8}T\cos\Delta f$$

$$C_{z4} = -C_{x5} \qquad C_{z5} = C_{x4}\,.$$